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Note

A Note on Bernstein–Durrmeyer Operators in $L_2(S)$

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The inequality

$$\omega_S^2(f, 1/\sqrt{n})_{L_2(S)} \leq C \|M_n f - f\|_{L_2(S)}$$

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In a recent paper, Berens, Schmid, and Xu [1] studied the rate of convergence of the Bernstein–Durrmeyer operator (given below by (4)) in terms of a measure of smoothness. They conjectured that

$$\omega_S^2(f, 1/\sqrt{n})_p \leq C \max_{k \geq n} \|M_k f - f\|_{L_p(S)}, \tag{1}$$

for $1 < p < \infty$ and proved that (1) is valid for $p = 2$. Recall that $\omega_S^r(f, t)_p$ (see [1, 4, Sect. 3; 6, Chap. 12, p. 202]) is given by

$$\omega_S^r(f, t)_p = \operatorname*{Sup}_{e \in V_S} \operatorname*{Sup}_{0 < h \leq t} \|\Delta^r_{h\varphi_e e} f\|_{L_p(S)}, \tag{2}$$

where V_S is the set of edges of S, $\varphi_{e_i}(x) = \sqrt{x_i(1 - |x|)}$ ($|x| = \sum_{i=1}^d x_i$), and $\varphi_{(e_i - e_j)/\sqrt{2}}(x) = \sqrt{x_i x_j}$. While we cannot prove (1) for $p \neq 2$, we present here a short proof of

$$\omega_S^2(f, 1/\sqrt{n})_2 \leq C \|M_n f - f\|_{L_2(S)} \tag{1'}$$

which is somewhat stronger than (1) for $p = 2$ as $\omega_S^2(f, t)$ is monotone.

The equivalence between the apropriate K-functional and $\omega_S^2(f, t)_2$ and the use of standard procedures make it clear that it is sufficient to show

$$\|P_{i,j}(D) M_n f\|_2 \leq \|P(D) M_n f\|_2 \leq n \|M_n f - f\|_2, \tag{3}$$

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where

$$P(D) \equiv \sum_{i \leq j} P_{i,j}(D) \equiv \sum_{i=1}^{d} \frac{\partial}{\partial x_i} x_i(1-|x|) \frac{\partial}{\partial x_i} + \sum_{i<j} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) x_i x_j \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right).$$

Derriennic [5, p. 157] proved that $\Pi_k = \Pi_{k-1} \oplus V_k$ where Π_k is the set of polynomials of total degree k and V_k is the eigenspace of $P(D)$ with the eigenvalue $-k(k+d)$. Derriennic [5, p. 157] also showed that

$$M_n f = \sum_{k=0}^{n} \lambda_{n,k} v_k(f), \qquad \lambda_{n,k} = \frac{(n+d)!\, n!}{(n+k+d)!\,(n-k)!}, \tag{4}$$

where $v_k(f)$ is the orthogonal projection of f on V_k. (Obviously, V_k is an eigenspace of M_n with eigenvalue $\lambda_{n,k}$.)

Now, after setting the notations, we are ready to prove (3).

The operator $P_{i,j}(D)$ is self adjoint and $P_{i,j}(D)\colon \Pi_k \to \Pi_k$. As $\langle P_{i,j}(D)\, v, u\rangle = \langle v, P_{i,j}(D)\, u\rangle = 0$ for $v \in V_k$ and $u \in \Pi_{k-1}$, we have $P_{i,j}(D)\, V_k \to V_k$. Hence, one can show $V_k = U_{0,k} \oplus \cdots \oplus U_{k,k}$ where $U_{l,k}$ is the eigenspace of $P_{i,j}(D)$ in V_k with eigenvalue $-l(l+1)$. This follows as $P_{i,j}(D)\, v = \lambda v$ for some polynomial v yields $\lambda = -l(l+1)$, l being the leading power of x_i in the expansion of v and the expansion is in $x_1, ..., x_d$ when dealing with $P_{i,i}(D)$ and in x_r for $r \neq j$ and $(1-|x|)$ in place of x_j for $P_{i,j}(D)$, $i \neq j$. We may now remark that $U_{l,k}$, constructed for the pair i, j, is an eigenspace of M_n, $P(D)$, and $P_{i,j}(D)$ and hence, on $U_{l,k}$, these operators commute. As the direct sum of $U_{l,k}$, $0 \leq k \leq m$, $0 \leq l \leq k$, is Π_m and since polynomials are dense in $W_p^l(S) \equiv \{f\colon (\partial/\partial\xi)^l f \in L_p(S)\}$, we have

$$M_n P_{i,j}(D) f = P_{i,j}(D)\, M_n f \quad \text{and} \quad P_{i,j}(D)\, P(D) f = P(D)\, P_{i,j}(D) f \tag{5}$$

in $W_p^2(S)$ and $W_p^4(S)$, respectively. (See [2, Lemma 2.5] for a similar argument.) Setting $u_{l,m}$ and v_m as the orthogonal projections of f on $U_{l,m}$ and V_m, respectively, we may use the Parseval equality to write

$$\begin{aligned} \|P_{i,j}(D)\, M_n f\|_2^2 &= \sum_{m=0}^{n} \sum_{l=0}^{m} (\lambda_{n,m} l(l+1))^2 \|u_{l,m}\|_2^2 \\ &\leq \sum_{m=0}^{n} \sum_{l=0}^{m} (\lambda_{n,m} m(m+d))^2 \|u_{l,m}\|_2^2 \\ &= \|P(D)\, M_n f\|_2^2. \end{aligned} \tag{6}$$

Using $(1-\lambda_{n,m})\,n \geqslant \lambda_{n,m} m(m+d)$ which can be proved by induction on m, we have

$$\|P(D)\, M_n f\|_2^2 = \sum_{m=0}^{n} (\lambda_{n,m} m(m+d))^2\, \|v_m\|_2^2$$
$$\leqslant \sum_{m=0}^{n} ((1-\lambda_{n,m})\, n)^2\, \|v_m\|_2^2 = n^2\, \|M_n f - f\|_2^2. \qquad (7)$$

The details of the induction proof are omitted. In [3]

$$\|P(D)\, M_n f\|_p \leqslant Cn\, \|M_n f - f\|_p, \qquad 1 < p < \infty,$$

which is weaker than (7) for $p = 2$, is proved.

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